Inhomogeneous scalar field solutions and inflation

S. E. Perez Bergliaffa [1] and K. E. Hibberd

Centro Brasileiro de Pesquisas Fisicas

Rua Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro - RJ

Brazil

Abstract

We present new exact cosmological inhomogeneous solutions for gravity coupled to a scalar field in a general framework specified by the parameter λ. The equations of motion (and consequently the solutions) in this framework correspond to either low-energy string theory or Weyl integrable spacetime according to the sign of λ. We show that different inflationary behaviours are possible, as suggested by the study of the violation of the strong energy condition. Finally, by the analysis of certain curvature scalars we found that some of the solutions may be nonsingular.

PACS numbers: 04.20.Jb 90.80.Cq 04.50+h

1 Introduction

The inflationary paradigm was first introduced by Guth [1] and solves, without the necessity of imposing highly special conditions on the initial state of the universe, some of the problems of the standard cosmological model. However, the homogeneity and isotropy problem still deserves some attention. This is due to the fact that a general proof of the naturalness of inflation is still to be derived. That is to say, it has not been shown that inflation will take place in any spacetime, independently of its symmetries. Since Gibbons and Hawking [2] and Gibbons and Moss [3] have stated the "cosmic no hair" conjecture, some progress has been made by Wald [4] in the case of Bianchi models. There has also been contributions from Jensen and Stein-Schabes [5], and Stein-Schabes and Barrow [6] for a class of inhomogeneous models which was quite close to spatial homogeneity. In all of these cases, inflation was driven by a positive cosmological constant as the conjecture requires.

In the more general case of a scalar field powering the expansion, Collins and Hawking [7] studied whether homogeneous models can approach isotropy. They found that only an insignificantly small number of these models can isotropize. Morover, there are some "no go" results for a large class of scalar field potentials [8]. Byland and Scialom [9] studied the issue of isotropization and inflation in Bianchi I, Bianchi III and Kantowski-Sachs models. They found that isotropy can be reached without inflation for Bianchi I and that if inflation takes place then isotropy is always reached, confirming previous results by Burd an Barrow [10].

In spite of these advances, whether or not inflation is generic is still an open issue either in the case of anisotropic or inhomogeneous scalar field universes. In particular for the inhomogeneous case some numerical and qualitative work has been done on the occurrence of inflation when initial inhomogeneities are present [11, 12, 13]. However, only a few exact solutions can be found in the literature. In a number of papers, Feinstein *et al* [14, 15, 16] showed that the behaviour of the inhomogeneous solutions can be diverse. In some cases their

[1]E-mail: santiago@lafex.cbpf.br

solutions inflate and homogeneize, but they do not isotropize [14]. In [15] it was shown that only some regions of spacetime undergo inflation, while for the geometries discussed in [16] inflation never occurs.

The scalar field which is a basic ingredient for inflation, arises naturally at in least three different contexts: Kaluza-Klein theories [17], low-energy string theory [18] and Weyl's theory in its integrable version (WIST) [19]. As we shall see later, a salient feature of WIST is that for a certain range of values of a parameter λ, the equations of motion coincide with those obtained in the low-energy limit of string theory in the so-called Einstein frame, neglecting all matter fields but the dilaton [2] [20]. Consequently, our aim is to derive and analyze the properties of inhomogeneous cosmological solutions of the equations of motion in the general setting of WIST. In so doing, we will obtain new solutions for low energy string theory in the case mentioned above and solutions with no counterpart in General Relativity (GR), *i.e.* unique to WIST.

In this work we shall use a model which generalizes Bianchi I cosmologies by the addition of some inhomogeneity in one dimension. These models are describable by the generalized Eistein-Rosen metric [28]. The origin of the inhomogeneity may be related to the presence of a background of primordial gravitational waves, which may have played a role in the early stages of the Universe [14]. The scalar field of the model is under the influence of a Liouville type potential, which arises as an effective potential in many unification theories [21]. Also this potential might give rise to a power law-inflation. It has been suggested by Müller *et al* [22] that this type of inflation is an attractor in the space of solutions of Einstein's equations in the limit $t \to \infty$ for the case of inhomogeneous spacetimes with a minimally coupled scalar field.

The structure of the paper is as follows. In Section 2 we give a short review of the basics of WIST integrable geometry and in Section 3 we derive the field equations. Sections 4 and 5 are devoted to two different types of solutions for the generalized Einstein-Rosen geometry. In Section 6 we discuss whether the solutions display inflation as indicated by the violation of the strong energy condition (SEC). We also study their singularity pattern by considering certain curvature scalars. Our conclusions and final remarks are given in Section 7.

2 Weyl integrable space-time

Weylian geometry is a generalization of Riemannian geometry in which the metric and a scalar field are the fundamental objects [3]. The difference between the two geometries stems from the law of parallel transport, which in the case of Weylian geometry is given by

$$g_{\mu\nu;\delta} = \omega_\delta g_{\mu\nu}. \tag{1}$$

The field ω_δ governs the variation of the standards of measure according to

$$dl = \frac{l}{2}\omega_\alpha dx^\alpha. \tag{2}$$

If we impose that the vector field ω_μ be a gradient of a certain field ω, that is,

$$\omega_\mu = \partial_\mu \omega, \tag{3}$$

[2] In turn, these are the equations for General Relativity plus a minimally coupled scalar field.

[3] A review of Weylian geometry can be found in [23].

then length variations along a closed path are integrable. The resulting geometrical structure is called Weyl Integrable Space-Time (WIST) [19]. We see therefore that in WIST the scalar field has a purely geometrical origin.

The simplest action we can write for this geometry in a vacuum is

$$S_W = \int (R + \xi \omega^{;\mu}_{;\mu}) \sqrt{-g}\, d^4x, \tag{4}$$

where R is the the scalar curvature for the Weylian geometry and ξ is an arbitrary coupling constant. The variation of S_W gives the following equations of motion

$$G_{\mu\nu} + \omega_{;\mu;\nu} - (2\xi - 1)\omega_{;\mu}\omega_{;\nu} + \xi g_{\mu\nu}\omega_{\ ;\alpha}\omega^{\ ;\alpha} = 0, \tag{5}$$

$$\omega^{;\alpha}_{;\alpha} + 2\omega_{;\alpha}\, \omega^{;\alpha} = 0. \tag{6}$$

These equations are written in terms of the tensors associated to the Weylian geometry. However they can be recast using the corresponding Riemannian tensors in the following way [23]:

$$\widehat{G}_{\mu\nu} = -\lambda(\omega_{||\mu}\omega_{||\nu} - \frac{1}{2} g_{\mu\nu}\omega^{||\alpha}\omega_{||\alpha}), \tag{7}$$

$$\widehat{\Box}\omega = 0, \tag{8}$$

where $\lambda = 3 - 4\xi$. According to the conventions we are using for the Riemann tensor, the source-like term coincides with the stress-energy tensor of a scalar field in a Riemannian spacetime only if the constant λ is negative. As was stated above, the solutions will depend on the parameter λ, which determines the type of theory we are dealing with. For a negative λ we recover GR, while $\lambda > 0$ provides us with a different sector of WIST. Some solutions of interest in the case of $\lambda < 0$ have been mentioned in the introduction. Several exact solutions with $\lambda > 0$ have been found in WIST. Among them, we can cite nonsingular cosmologies [19], static and spherically symmetric spacetimes [24], inflationary solutions [25], and dilaton electromagnetic Bianchi-I cosmologies [26]. The coupling of different types of matter to the geometry of WIST was studied in [27]. In the next section we will derive the expression for the field equations (7) and (8) in the generalized Einstein-Rosen metric.

3 Field equations

In what follows, we shall work with one-dimensional inhomogeneities which are describable by the generalized Einstein-Rosen metric. These spacetimes admit an Abelian group of isometries G_2 and include Bianchi models of types I-VII [28]. Assuming that the two Killing vectors are hypersurface orthogonal, the metric of the model is given by the following diagonal form:

$$ds^2 \;=\; e^{f(t,z)}(-dt^2 + dz^2) + G(t,z)[e^{h(t,z)}dx^2 + e^{-h(t,z)}dy^2]. \tag{9}$$

The potential is taken as

$$V(\omega) = \Lambda e^{-\lambda k \omega(t,z)}, \tag{10}$$

where Λ and k are arbitrary constants. The field equations for the geometry are

$$\frac{\ddot{G}}{G} - \frac{G''}{G} = 2e^f V, \tag{11}$$

$$\frac{\ddot{G}}{G} + \frac{G''}{G} - \frac{1}{2}\left(\frac{\dot{G}}{G}\right)^2 - \frac{1}{2}\left(\frac{G'}{G}\right)^2 - \frac{\dot{f}\dot{G}}{G} - \frac{f'G'}{G} + \frac{1}{2}\dot{h}^2 + \frac{1}{2}h'^2 = \lambda(\dot{\omega}^2 + \omega'^2), \tag{12}$$

$$\frac{\dot{G}'}{G} - \frac{\dot{G}G'}{2G^2} + \frac{1}{2}\dot{h}h' - \frac{f'\dot{G}}{2G} - \frac{\dot{f}G'}{2G} = \lambda\dot{\omega}\omega', \tag{13}$$

$$\ddot{h} - h'' + \frac{\dot{G}\dot{h}}{G} - \frac{G'h'}{G} = 0, \tag{14}$$

$$\ddot{\omega} - \omega'' + \frac{\dot{G}\dot{\omega}}{G} - \frac{G'\omega'}{G} - \frac{e^f}{\lambda}\frac{\partial V}{\partial \omega} = 0. \tag{15}$$

We shall find particular solutions to these equations for two different cases: $G = G(t)$ (which corresponds to the case in which the element of the transitivity surface is homogeneous) and $G = G(t, z)$. To solve the equations we shall adapt the method presented in [14] for the first case and the steps given in [15] for the second case.

4 G=G(t)

To solve Eqs.(11)-(15), we write the scalar field as follows [14],

$$\omega(t, z) = -\frac{k}{2}\ln G(t) + \Omega(t, z). \tag{16}$$

After substituting Eq.(16) into Eq.(15) we obtain an expression for $\Omega(t, z)$,

$$\ddot{\Omega} - \Omega'' + \frac{\dot{G}}{G}\dot{\Omega} = 0. \tag{17}$$

Suppose that the functions $h(t, z)$ and $\Omega(t, z)$ can be split as

$$h(t, z) = \Pi(t) + P(z), \qquad \Omega(t, z) = \chi(t) + \psi(z). \tag{18}$$

Using Eq.(18) in Eqs.(14) and (17) we get

$$h(t, z) = \Pi(t) + \frac{B}{2}z^2 + Cz, \qquad \Phi(t, z) = \chi(t) + \frac{E}{2}z^2 + Fz, \tag{19}$$

where B, C, E and F are arbitrary constants of integration. Substituting this equation and the expression for $f(t, z)'$ obtained by differentiating Eq.(11) into Eq.(13) we obtain

$$\frac{1}{2}\dot{\Pi}(Bz + C) - \lambda\dot{\chi}(Ez + F) = 0, \tag{20}$$

where λ is the constant that appears in the equations of motion. From Eqs.(11), (15) and (19) we arrive at

$$\frac{\ddot{G}}{G} - \frac{\dddot{G}}{\ddot{G}}\frac{\dot{G}}{G} - K\left(\frac{\dot{G}}{G}\right)^2 + \frac{1}{2}\dot{\Pi}^2 - \lambda\dot{\chi}^2 + \frac{1}{2}(Bz + C)^2 - \lambda(Ez + F)^2 = 0, \tag{21}$$

with $K = -\frac{1}{4}(\lambda k^2 + 2)$. It follows from this equation and Eq.(20) that

$$\dot{\Pi} = \frac{2\lambda F}{C}\dot{\chi}. \tag{22}$$

Eq.(14) now gives

$$\dot{\Pi} = \frac{A}{G}, \tag{23}$$

where A is a constant.

Finally, after substituting Eqs.(22) and (23) into Eq.(21) we obtain an equation that governs the evolution of the function $G(t)$:

$$\ddot{G}^2 G - \dddot{G}\,\dot{G}G - K\ddot{G}\dot{G}^2 + M^2\ddot{G} + N^2 G^2 \ddot{G} = 0, \tag{24}$$

where the constants M are N are given by

$$M^2 = \frac{A^2}{2}\left(1 - \frac{C^2}{2\lambda F^2}\right), \qquad N^2 = \left(\frac{C^2}{2} - \lambda F^2\right). \tag{25}$$

Also the functions appearing in the metric can be rewritten as the following

$$h(t,z) = A\int \frac{dt}{G} + Cz, \tag{26}$$

$$\omega(t,z) = -\frac{k}{2}\ln G + \frac{CA}{2F\lambda}\int \frac{dt}{G} + Fz, \tag{27}$$

$$f(t,z) = \lambda k\omega + \ln\frac{\ddot{G}}{G} - \ln 2\Lambda. \tag{28}$$

Therefore any solution of Eq.(24) will determine the functions in the metric (9).

Next we present particular cases of this geometry. Each case is associated to a different choice of the function $G(t)$. A summary of the solutions and constraints on the parameters is displayed in Table 1 for $\lambda > 0$.

Let us remark that all of the solutions with $\lambda > 0$ are new. In all four cases, the possible values of the parameter λ are restricted by the integration constants appearing in the solution. Note that for $G(t) = t^\beta$, the function $f(t,z)$ contains a logarithmic term involving constants which will be important when studying the possible violation of the SEC (see Section 6).

There exist similar solutions in the case of GR plus a minimally coupled scalar field. Note that in the first two solutions of Table 1, only $f(t,z)$ depends on λ. The solutions for the cases $G(t) = \alpha\sinh\beta t$ and $G(t) = \alpha\sin\beta t$ in GR are easily obtained by setting $\lambda = -1$ in the corresponding expressions in Table 1. They are only subjected to the constraint $k^2 > 2$. These results are summarized in Table 2.

For the solutions obtained in GR, we shall point out that $G(t) = t^\beta$ and $G(t) = \alpha\sin\beta t$ are new and have not been discussed previously in the literature. The other two solutions we were able to derive are more general than those obtained in [14].

5 $G = G(t,z)$

As in the case for $G = G(t)$, we assume that ω may be written in the form [15]

$$\omega(t,z) = -\frac{k}{2}\ln G(t,z) + \Omega(t,z). \tag{29}$$

Using Eq.(15) one may obtain the relation

$$\ddot{\Omega} - \Omega'' + \frac{\dot{G}}{G}\dot{\Omega} - \frac{G'}{G}\Omega' = 0. \tag{30}$$

$G(t)$	Constraints	Functions
$e^{\beta t}$	$\beta^2 = 2\ \frac{2\lambda F^2 - C^2}{\lambda k^2+2}$, $\lambda > \frac{C^2}{2F^2}$, $A = 0\ \ , \Lambda > 0$,	$h(t,z) = Cz$, $\omega(t,z) = -\frac{k}{2}\beta t + Fz$, $f(t,z) = -\frac{\lambda k^2}{2}\beta t + \lambda k F z + \ln\left(\frac{\beta^2}{2\Lambda}\right)$.
t^{β}	$\beta = -2\ \frac{1\pm\sqrt{1+2(\lambda k^2+2)(2\lambda F^2-C^2)}}{\lambda k^2+2}$, $\lambda > \frac{C^2}{2F^2}$, $A = 0$,	$h(t,z) = Cz$, $\omega(t,z) = -\frac{k}{2}\beta \ln t + Fz$, $f(t,z) = -\left(\frac{\lambda k^2}{2}\beta + 2\right)\ln t + \lambda k F z + \ln\left[\frac{\beta(\beta-1)}{2\Lambda}\right]$.
$\alpha \sinh \beta t$	$\beta^2 = 2\ \frac{2\lambda F^2 - C^2}{\lambda k^2+2}$, $\alpha^2 = \frac{A^2}{2\lambda F^2}\ \frac{\lambda k^2+2}{2-\lambda k^2}$, $\lambda \in \left(\frac{C^2}{2F^2}, \frac{2}{k^2}\right)$, $\delta = \frac{AC}{2\lambda F\alpha\beta}$, $\Lambda > 0$,	$h(t,z) = \frac{A}{\alpha\beta}\ln\left(\tanh\frac{\beta t}{2}\right) + Cz$, $\omega(t,z) = -\frac{k}{2}\ln\alpha + \ln\left[\frac{(\tanh\frac{\beta t}{2})^{\delta}}{(\sinh\beta t)^{\frac{k}{2}}}\right] + Fz$, $f(t,z) = \ln\left(\frac{\beta^2\alpha^{-\lambda k^2/2}}{2\Lambda}\right) + \ln\left[\frac{(\tanh\frac{\beta t}{2})^{\delta\lambda k}}{(\sinh\beta t)^{\lambda k^2/2}}\right] + \lambda k F z$.
$\alpha \sin \beta t$	$\beta^2 = -2\ \frac{2\lambda F^2 - C^2}{\lambda k^2+2}$, $\alpha^2 = \frac{A^2(\lambda k^2+2)}{2\lambda F^2(\lambda k^2-2)}$, $\lambda \in \left(\frac{2}{k^2}, \frac{C^2}{2F^2}\right)$, $\delta = \frac{AC}{2\lambda F\alpha\beta}$, $\Lambda > 0$,	$h(t,z) = \frac{A}{\alpha\beta}\ln\left(\tan\frac{\beta t}{2}\right) + Cz$, $\omega(t,z) = -\frac{k}{2}\ln\alpha + \ln\left[\frac{(\tan\frac{\beta t}{2})^{\delta}}{(\sin\beta t)^{\frac{k}{2}}}\right] + Fz$, $f(t,z) = \ln\left(\frac{\beta^2\alpha^{-\lambda k^2/2}}{2\Lambda}\right) + \ln\left[\frac{(\tan\frac{\beta t}{2})^{\delta\lambda k}}{(\sin\beta t)^{\lambda k^2/2}}\right] + \lambda k F z$.

Table 1: Particular solutions to Eqns.(11)-(15) for $\lambda > 0$.

$G(t)$	Constraints	Functions
$e^{\beta t}$	$\beta^2 = 2\ \frac{2F^2+C^2}{k^2-2}$, $A = 0\ \ , \Lambda > 0$, $k^2 > 2$,	$f(t,z) = \frac{k^2}{2}\beta t - kFz + \ln\left(\frac{\beta^2}{2\Lambda}\right)$.
t^{β}	$\beta = -2\ \frac{1\pm\sqrt{1-2(2-k^2)(2F^2+C^2)}}{k^2-2}$, $k^2 > 2 - \frac{1}{2(2F^2+C^2)}$, $C^2 > \frac{1}{4} - 2F^2$, $A = 0$,	$f(t,z) = \left(\frac{k^2\beta}{2} - 2\right)\ln t - kFz + \ln\left[\frac{\beta(\beta-1)}{2\Lambda}\right]$.,

Table 2: Particular cases of the geometry (9) for $\lambda = -1$.

This equation together with Eq.(14), implies the following general form for $\omega(t,z)$

$$\omega(t,z) = -\frac{k}{2}\ln G(t,z) + mh(t,z), \tag{31}$$

where m is a constant. We further assume that the function $G(t,z)$ is separable:

$$G(t,z) = T(t)Z(z). \tag{32}$$

Eqs.(13)-(15) suggest that two *Ansätze* are possible for h:

$$e^{h(t,z)} = Q(t)Z(z)^n, \tag{33}$$

or

$$e^{h(t,z)} = T(t)^n P(z). \tag{34}$$

These two cases will be treated separately below.

5.1 Case I: $e^{h(t,z)} = Q(t)Z(z)^n$

After substituting Eqn.(33), Eq.(14) takes the form

$$n\frac{Z''}{Z} = \frac{\dot{T}\dot{Q}}{TQ} + \frac{\ddot{Q}}{Q} - \frac{\dot{Q}^2}{Q^2} = n\epsilon a^2, \tag{35}$$

with $\epsilon = 0, \pm 1$. The spatial part of this equation can be easily solved and leads to

$$Z(z) = \begin{cases} A\cosh az + B\sinh az, & \epsilon = 1, \\ Az + B, & \epsilon = 0, \\ A\cos az + B\sin az, & \epsilon = -1. \end{cases} \tag{36}$$

The temporal part of Eq.(35) can be written as an integral equation which will be given below (Eq.(41)). Eq.(11) will be taken as the definition of the function $f(t,z)$. Also note that we have not yet used Eqs.(12) and (13). They can be rewritten in a more useful way, as we shall present below (Eqs.(42) and (43)). These considerations lead us to the following system of equations:

$$\begin{aligned} G(t,z) &= T(t)Z(z), && (37)\\ h(t,z) &= \ln Q(t) + n\ln Z(z), && (38)\\ \omega(t,z) &= -\frac{k}{2}\ln[T(t)Z(z)] + mh(t,z), && (39)\\ f(t,z) &= \lambda k\omega(t,z) + \ln\left(\frac{\ddot{T}}{T} - \frac{Z''}{Z}\right) - \ln 2\Lambda, && (40)\\ \ln Q(t) &= n\epsilon a^2 \int \frac{\left(\int_0^t T(\tau)d\tau\right)}{T(t)}dt, && (41) \end{aligned}$$

$$\frac{f'Z}{Z'} = \frac{T}{\dot{T}}\left[-\dot{f} + \frac{\dot{T}}{T}\left(1 - \frac{\lambda k^2}{2} + \lambda mnk\right) + \frac{\dot{Q}}{Q}(n - 2\lambda m^2 n + \lambda mk)\right] \tag{42}$$

$$\begin{aligned} -\frac{Z''}{Z} - \frac{Z'^2}{Z^2}\left(-\lambda m^2 n^2 + \lambda mnk - \frac{\lambda k^2}{4}\right. &- \left.\frac{1}{2} + \frac{n^2}{2}\right) + \frac{f'Z'}{Z} \\ &= \frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}\left(\frac{\lambda k^2}{4} + \frac{1}{2}\right) - \frac{\dot{f}\dot{T}}{T} + \frac{\dot{Q}^2}{Q^2}\left(\frac{1}{2} - \lambda m^2\right) + \lambda mk\frac{\dot{Q}\dot{T}}{QT}. \end{aligned} \tag{43}$$

We seek particular solutions for this system.

Eqn.(43) suggests the following convenient form of $T(t)$,

$$T(t) = e^{\beta t}. \tag{44}$$

From Eq.(41) we obtain the corresponding $Q(t)$,

$$Q(t) = \exp\left(\frac{n\epsilon a^2}{\beta} t\right), \tag{45}$$

The solution is specified by $T(t)$, $Q(t)$ and $Z(z)$ and Eqs.(42) and (43) give constraints for the different constants that appear in the solution. According to the choice of $Z(z)$, we shall study three different cases. Table 3 summarizes the solutions obtained with the above values for $T(t)$ and $Q(t)$, with $\lambda > 0$.

It can be shown from Eqns.(42) and (43) that $Z(z) = Az$ is not a solution for $\lambda > 0$ for the choices (44), $T(t) = \sin\beta t$, $T(t) = \cos\beta t$, and $T(t) = \sinh\beta t$. It is however a solution in GR [15]. The two solutions displayed on Table 3 are new for $\lambda > 0$ and they were also shown to be solutions in GR in [15].

5.2 Case II: $e^{h(t,z)} = T(t)^n P(z)$

In analogy with the previous case, we arrive at the following system of equations:

$$\begin{aligned}
G(t,z) &= T(t)Z(z), && (46)\\
h(t,z) &= n\ln T(t) + \ln P(z), && (47)\\
\omega(t,z) &= -\frac{k}{2}\ln[T(t)Z(z)] + mh(t,z), && (48)\\
f(t,z) &= \lambda k\omega(t,z) + \ln\left(\frac{\ddot{T}}{T} - \frac{Z''}{Z}\right) - \ln 2\Lambda, && (49)\\
\ln P(z) &= n\epsilon a^2 \int \frac{\left(\int_0^z Z(\eta)d\eta\right)}{Z(z)} dz, && (50)
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{f}T}{\dot{T}} &= \frac{Z}{Z'}\left[-f' + \frac{Z'}{Z}\left(1 - \frac{\lambda k^2}{2} + \lambda mnk\right) + \frac{P'}{P}(n - 2\lambda m^2 n + \lambda mk)\right] && (51)\\
-\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}\left(-\lambda m^2 n^2 + \lambda mnk - \frac{\lambda k^2}{4}\right. &- \left.\frac{1}{2} + \frac{n^2}{2}\right) + \frac{f'Z'}{Z}\\
&= \frac{Z''}{Z} - \frac{Z'^2}{Z^2}\left(\frac{\lambda k^2}{4} + \frac{1}{2}\right) - \frac{f'Z'}{Z} + \frac{P'^2}{P^2}\left(\frac{1}{2} - \lambda m^2\right) + \lambda mk\frac{Z'P'}{ZP} && (52)
\end{aligned}$$

Table 3 summarizes the solutions obtained with

$$Z(z) = e^{\beta z}, \quad P(z) = \exp\left(\frac{n\epsilon a^2}{\beta} z\right). \tag{53}$$

Let us mention that for $\lambda > 0$ the case $\epsilon = 0$ is not a solution and that both solutions presented for $T(t)$ are new. For the value $\lambda = -1$, we recover the corresponding solutions in GR obtained by [15].

6 Inflation and singularities

The presence of inflation is distinctly signalled out by the behaviour of the deceleration factor q. In order to calculate q, we need to define a velocity field u_μ for the matter in such a way that $u_\mu u^\mu = -1$. The usual

$Z(z)$	Constraints	Solutions
$\sin az$	$\beta=\pm a\sqrt{\frac{2-\lambda k^2}{2+\lambda k^2}}$, $m=\pm\frac{\sqrt{\lambda(\lambda k^2+2n^2-2)}}{2n\lambda}$, $n^2>\frac{2-\lambda k^2}{2}$, $k^2<\frac{2}{\lambda}$, $\Lambda>0$,	$G(t,z)=e^{\beta t}\sin az$, $h(t,z)=-\frac{na^2}{\beta}t+n\ln(\sin az)$, $\omega(t,z)=-\left(\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)t+\left(mn-\frac{k}{2}\right)\ln(\sin az)$, $f(t,z)=-\lambda k\left(\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)t+\lambda k\left(mn-\frac{k}{2}\right)\ln(\sin az)+\ln\left(\frac{\beta^2+a^2}{2\Lambda}\right)$.
$\sinh az$	$\beta=\pm a\sqrt{\frac{\lambda k^2-2}{\lambda k^2+2}}$, $m=\pm\frac{\sqrt{\lambda(\lambda k^2+2n^2-2)}}{2n\lambda}$, $n^2>\frac{2-\lambda k^2}{2}$, $k^2>\frac{2}{\lambda}$, $\Lambda<0$,	$G(t,z)=e^{\beta t}\sinh az$, $h(t,z)=\frac{na^2}{\beta}t+n\ln(\sinh az)$, $\omega(t,z)=\left(-\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)t+\left(mn-\frac{k}{2}\right)\ln(\sinh az)$, $f(t,z)=\lambda k\left(-\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)t+\lambda k\left(mn-\frac{k}{2}\right)\ln(\sinh az)+\ln\left(\frac{\beta^2-a^2}{2\Lambda}\right)$.

Table 3: Particular solutions of the system given by Eqs.(37)-(43), using Eqs.(44) and (45).

$T(t)$	Constraints	Solutions
$\sin at$	$\beta=\pm a\sqrt{\frac{2-\lambda k^2}{2+\lambda k^2}}$, $m=\pm\frac{\sqrt{\lambda(\lambda k^2+2n^2-2)}}{2n\lambda}$, $n^2>\frac{2-\lambda k^2}{2}$, $k^2<\frac{2}{\lambda}$, $\Lambda<0$,	$G(t,z)=e^{\beta z}\sin at$, $h(t,z)=-\frac{na^2}{\beta}z+n\ln(\sin at)$, $\omega(t,z)=-\left(\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)z+\left(mn-\frac{k}{2}\right)\ln(\sin at)$, $f(t,z)=-\lambda k\left(\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)z+\lambda k\left(mn-\frac{k}{2}\right)\ln(\sin at)+\ln\left(\frac{-\beta^2-a^2}{2\Lambda}\right)$.
$\sinh at$	$\beta=\pm a\sqrt{\frac{\lambda k^2-2}{\lambda k^2+2}}$, $m=\pm\frac{\sqrt{\lambda(\lambda k^2+2n^2-2)}}{2n\lambda}$, $n^2>\frac{2-\lambda k^2}{2}$, $k^2>\frac{2}{\lambda}$, $\Lambda>0$,	$G(t,z)=e^{\beta z}\sinh at$, $h(t,z)=\frac{na^2}{\beta}z+n\ln(\sinh at)$, $\omega(t,z)=\left(-\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)z+\left(mn-\frac{k}{2}\right)\ln(\sinh az)$, $f(t,z)=\lambda k\left(-\frac{k\beta}{2}+\frac{mna^2}{\beta}\right)z+\lambda k\left(mn-\frac{k}{2}\right)\ln(\sinh at)+\ln\left(\frac{a^2-\beta^2}{2\Lambda}\right)$.

Table 4: Particular solutions to the system given by Eqs.(46)-(52), with Eqns.(53).

definition of a four-velocity orthogonal to the hypersurfaces $\omega =$ const. given by

$$v_\alpha = \frac{\omega_\alpha}{\sqrt{-\omega_\alpha \omega^\alpha}} \tag{54}$$

is not suitable because for some of the cases studied here the gradient of ω is spacelike. We shall follow instead a different route. The nonfulfillment of the SEC is known to be a necessary condition for inflation to occur. Therefore, we shall study the behaviour of the quantity $\tau = (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) u^\mu u^\nu$ (where T is the trace of the energy-momentum tensor). Whenever this quantity is negative or zero, SEC is violated and inflation may be possible.

We also would like to make some statements regarding the (absence of) singularities that these geometries may display. The behaviour of the invariants $R_1 = g_{\mu\nu} R^{\mu\nu}$, $R_2 = R_{\mu\nu} R^{\mu\nu}$, $R_3 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ is of help in this task. The results are tabulated in Table 5, including the analysis of τ for each case, for a positive λ.

From Table 5 we see that for solution (1) the entire spacetime may undergo inflation for $\Lambda > 0$. In the analogous solution of GR however, inflation requires there that $k \in (-2, 2)$ [14], but this is actually forbidden even in the case $\beta \neq 1$. This can be seen from the definition of β^2 for this solution given in Table 2. Also, the calculation of the invariants show that this solution may be nonsingular both in GR and WIST (except perhaps for some special values of λ).

The case $G(t) = t^\beta$ must be treated separately. It is apparent that there may be inflation if $\beta > 1$, with $\lambda > 0$. Then the constraint imposed by the argument of the logarithm in the solution (see Table 1) is automatically satisfied. From this restriction, we can obtain an inequality that limits the possible values for λ. The general case is very involved, so we restrict our study to the special case $F = \pm \frac{k}{4}$. For these values, λ must fulfill

$$\lambda > 2 \ \frac{4C^2 + 3}{k^2 [1 - 4(C+1)]}, \tag{55}$$

with $C < -\frac{3}{4}$. The scalars R_1, R_2, and R_3 are regular for every t and z, but we have to impose again $\beta > 1$, which was a condition for inflation.

For the corresponding solution in GR, we may have inflation for at least $\Lambda > 0$ and $\beta < 0$. Also in the GR case, singularities may be absent if $\beta < 0$.

For solutions (3) and (4) of Table 5, the expression of τ shows that SEC may be violated only for some intervals of the t coordinate. Because the expression for the scalars is very involved, we can only state that they may be regular for special values of the constants.

In solution (5), the quantity between brackets in τ is always positive for $\lambda > 0$. However we expect that in the general case the sign of τ depends periodically on z [4]. The same situation occurs for solution (6), for at least $n = k/2m$. In both solutions, R_1, R_2 and R_3 are finite for all values of t but they may diverge for some z.

Finally, for solution (7) SEC may be violated only for a finite range of t values while in solution (8), τ is always negative for any values of t and z. In both cases, the scalars are finite for all z, but they may diverge for some t.

[4] The behaviour of solutions (5)-(8) for $\lambda < 0$ has been studied in [15].

G	SEC Violation	R_1, R_2, R_3
(1) $e^{\beta t}$	$\tau = -\frac{\Lambda}{2}(\lambda k^2+4)\exp\left[-\frac{\lambda k}{2}(-k\beta t+2Fz)\right]$ negative for $\Lambda > 0$.	regular for every t and z.
(2) t^{β}	$\tau = -\Lambda\left(\frac{\beta(\lambda k^2+4)-4}{2(\beta-1)}\right) e^{\lambda kFz}\ t^{\frac{\lambda k^2\beta}{2}}$ negative for $\beta > 1$ and $\Lambda > 0$.	regular for every t and z, with $\beta > 1$.
(3) $\alpha\sin\beta t$	$\tau = -\gamma^{-1}\left(\tan\frac{\beta t}{2}\right)^{\delta\lambda k}(\sin\beta t)^{\lambda k^2/2-2}e^{-\lambda Fz}\times$ $\left[\left(\frac{\lambda\beta^2k^2}{4}-2\Lambda\gamma\right)\cos^2\beta t-\lambda\beta^2\delta k\cos\beta t+\lambda\beta^2\delta^2+2\Lambda\gamma\right]$ negative for some interval of t.	complicated functions of t and z.
(4) $\alpha\sinh\beta t$	$\tau = \frac{1}{4\gamma}e^{-\lambda kFz}(\sinh\beta t)^{\lambda k^2/2+2}\left(\tanh\frac{\beta t}{2}\right)^{\delta\lambda k}\times$ $\left[-\frac{1}{4}(\lambda\beta^2k^2+\delta\Lambda\gamma)\cosh^2\beta t+\lambda\beta^2k\delta\cosh\beta t-\delta^2\beta^2\lambda+2\Lambda\gamma\right]$ negative for some interval of t.	complicated functions of t and z
(5) $e^{\beta t}\sin az$	$\tau = -\frac{\Lambda}{2\beta^2(a^2+\beta^2)}\exp\left[\frac{\lambda k(2mna^2+k\beta^2)}{2\beta}t\right]\times$ $(\sin az)^{\left[\lambda k\left(\frac{k}{2}-mn\right)\right]}[\lambda(k\beta^2+2mna^2)^2+4\beta^2(a^2+\beta^2)]$.	finite for all t.
(6) $e^{(\beta t)}\sinh az$	$\tau = -\frac{\Lambda}{2\beta^2(\beta^2-a^2)}\exp\left[-\frac{\lambda k(2mna^2-k\beta^2)}{2\beta}t\right]\times$ $(\sinh az)^{\left[\lambda k\left(\frac{k}{2}-mn\right)\right]}[\lambda(4mna^2(mna^2-k\beta^2)+k^2\beta^4)$ $+4\beta^2(\beta^2-a^2)]$.	finite for all t.
(7) $e^{\beta z}\sin at$	$\tau = \frac{\Lambda}{2(\beta^2+a^2)}\exp\left[\frac{\lambda k(-2mna^2+k\beta^2)}{2\beta}z\right]\times$ $(\sin at)^{\left[\lambda k\left(\frac{k}{2}-mn\right)-2\right]}\left[\lambda a^2(k-2mn)^2\cos^2 at-4(\beta^2+a^2)\sin^2 at\right]$.	finite for all z.
(8) $e^{\beta z}\sinh at$	$\tau = \frac{\Lambda}{2(\beta^2-a^2)}\exp\left[\frac{\lambda k(-2mna^2+k\beta^2)}{2\beta}z\right]\times$ $(\sinh at)^{\left[\lambda k\left(\frac{k}{2}-mn\right)-2\right]}$ $\left[\lambda a^2(k-2mn)^2\cosh^2 at-4(\beta^2-a^2)\sinh^2 at\right]$.	finite for all z.

Table 5: The study of the violation of the strong energy condition and curvature scalars.

7 Conclusions

We have studied different classes of G_2 inhomogeneous spacetimes for Weyl integrable spacetime (WIST). We were able to investigate simultaneously two types of solutions: those belonging exclusively to WIST (*i.e.* for $\lambda > 0$) and those that are solutions of GR (or low energy string theory, when considering solely the dilaton and the graviton). In the two cases the scalar field was under the influence of an exponential potential. The eight solutions with positive λ presented in Tables 1, 3, and 4 are new.

Setting $\lambda = -1$, WIST reduces to GR plus a scalar field. Using this fact, we were also able to obtain two new solutions in GR. By the inclusion of extra constants, two other solutions obtained by [14] were generalized.

These solutions were summarized in Table 2.

The results presented in Table 5 indicate that for all of the solutions determined by $G = G(t)$ with $\lambda > 0$ (see Table 1), SEC is violated and consequently there may be inflation. In particular, for solutions (1) and (2) inflation is global, while for (3) and (4) the violation if any, depends on the t coordinate. We also noted that while solution (1) may be inflationary for $\lambda > 0$, there is no inflation in the GR case. However for the other solutions (see Table 2) there is inflation for negative λ.

The second group of solutions characterized by $G = G(t, z)$, is divided into two subgroups (see Tables 3 and 4). Our results show that the inflationary behaviour of all these solutions is similar to the corresponding solutions in GR obtained by Feinstein *et al* [15]. In the first subgroup summarized in Table 3, there is a strong spatial dependence and SEC violation is more probable in some regions (for instance, near the hypersurface $z = 0$ in the hyperbolic case). In the second subgroup described in Table 4, the dependence on t dominates leading to analogous conclusions.

In summary, all of the solutions we have presented in this article exhibit inflation either globally or for certain regions of spacetime and so their behaviour agrees with the conjecture of the naturalness of inflation.

We have also studied (whenever possible) the occurrence of singularities. We found that some of the models are nonsingular for any value of λ when particular constraints on the integration constants are fulfilled. This fact, which was not discussed in the GR case of [14], may be understood in terms of Raychaudhuri's equation:

$$\theta_{,\mu} v^{\mu} = \dot{v}^{\mu}_{;\mu} + 2\Omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 + \lambda(\omega_{,\mu} v^{\mu})^2 \tag{56}$$

Note that $\Omega^2 = 0$ in all of our solutions and that the last term of this equation is always negative in GR. From this we may conclude that solutions (1) and (2) in Table 5 taken in the limit $\lambda = -1$ have a nonzero acceleration. In the case $\lambda > 0$ the situation is different. The singularity may be avoided by the combined effect of the acceleration and scalar field terms.

Finally we would like to remark that dynamical analysis techniques applied to the equations of motion of WIST (written in the appropiate set of variables) should permit a study of the asymptotic behaviour of these models for any value of λ [29]. Based on this type of analysis for a homogeneous and isotropic solution performed by Oliveira *et al* [30], we expect that the behaviour of general G_2 inhomogeneous cosmologies with $\lambda > 0$ will be different to the GR case. We hope to report more on this issue in a future publication.

8 Acknowledgements

SEPB would like to acknowledge financial support from CLAF and CONICET-Argentina. KEH would like to acknowledge financial support from CNPq-Brazil. Both authors are grateful to J. Salim, M. Novello and I. Soares for useful comments.

References

[1] A. Guth, Phys. Rev. D **23**, 347 (1981).

[2] G. Gibbons and S. Hawking, Phys. Rev. D **15**, 2738 (1977).

[3] S. Hawking and I. Moss, Phys. Lett. B **110**, 35 (1982).

[4] R. Wald, Phys. Rev. D **28**, 2188 (1983).

[5] L. Jensen and J. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).

[6] J. Stein-Schabes and J. Barrow, Phys. Lett. A **103**, 3115 (1984).

[7] C. B. Collins and S. W. Hawking, Astrophys. J. **180**, 317 (1973).

[8] M. Heusler, Phys. Lett. B **253**, 33 (1991).

[9] S. Byland and D. Scialom, Phys. Rev. D **98**, 6065 (1998).

[10] A. Burd and J. Barrow, Nucl. Phys. B **308**, 929 (1988).

[11] O. Iguchi and H. Ishihara, Phys. Rev. D **56**, 3216 (1997).

[12] H. Kurki-Suonio, J. Centrella, R. Matzner, and J. Wilson, Phys. Rev. D **35**, 435 (1987).

[13] D. Goldwirht and T. Piran, Phys. Rev. D **40**, 3263 (1989).

[14] J. M. Aguirregabiria, A. Feinstein and J. Ibañez, Phys. Rev. D **48**, 4669 (1993).

[15] A. Feinstein, J. Ibañez and P. Labraga, J. Math. Phys. **36**, 4962 (1995).

[16] A. Feinstein and J. Ibañez, Class. Q. Grav. **10**, L227 (1993).

[17] A. Love and D. Bailin, Rep. Prog. Phys. **50**, 1087 (1987).

[18] M. Green, J. Schwartz and E. Witten, *"Superstring theory,"* (Cambridge U. Press, 1987).

[19] M. Novello, L. A. Oliveira, J. M. Salim and E.Elbaz, Int. J. Mod. Phys. D **1**, 641 (1993).

[20] See for instance J. Barrow and K. Kunze, invited paper to appear in the special issue of the Journal of Chaos, Solitons and Fractals on "Superstrings, M, F, S... Theory", eds. C. Castro and M.S. El Naschie, and gr-qc/9807040.

[21] J. Halliwell, Phys. Lett. B **185**, 341 (1987).

[22] V. Müller, H. Schmidt and A. Starobinsky, Class. Quantum Grav. **7**, 1163 (1990).

[23] J. M. Salim and S. L. Sautu, Class. Quantum Grav. **13**, 353 (1996).

[24] J. Salim and S. Sautu, Class. Quantum Grav. **15**, 203 (1998).

[25] J. Fabris, J. Salim and S. Sautu, Mod. Phys. Lett. A **13**, 953 (1998).

[26] J. Salim, S. Sautu, and R. Martins, Clas. Quantum Grav. **15**, 1521 (1998).

[27] J. Salim and S. Sautu, Class. Quantum Grav. **13**, 353 (1996).

[28] M. Carmeli, Ch. Carach and S. Malin, Phys. Rep. **76**, 79 (1981).

[29] J. Ibañez and I. Olasagasti, Class. Quantum Grav. **15**, 1937 (1998).

[30] H. de Oliveira, J. Salim and S. Sautu, Class. Quantum Grav. **14**, 2833 (1997).